J.K. SHAH CLASSES

MATHEMATICS & STATISTICS

SYJC TEST - 01

TOPIC : CONTINUITY + DIFFERENTIATION | **DURATION - 1 $^{1}/2$ HR** | **MARKS - 30**

SOLUTION SET

Q1. Attempt any FIVE of the following (2 marks each) **(10 marks)**

Q1

01. $f(x) = \dfrac{\sqrt{4+x} - 2}{3x}$; $x \neq 0$

$= 1/4$; $x = 0$ Discuss continuity at $x = 0$

Solution :

Step 1

$\lim\limits_{x \to 0} f(x)$

$= \lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{3x}$

$= \lim\limits_{x \to 0} \dfrac{\sqrt{4+x} - 2}{3x} \cdot \dfrac{\sqrt{4+x} + 2}{\sqrt{4+x} + 2}$

$= \lim\limits_{x \to 0} \dfrac{4 + x - 4}{3x} \cdot \dfrac{1}{\sqrt{4+x} + 2}$

$= \lim\limits_{x \to 0} \dfrac{\cancel{x}}{3\cancel{x}} \cdot \dfrac{1}{\sqrt{4+x} + 2}$ $x \neq 0$

$= \lim\limits_{x \to 0} \dfrac{1}{3} \cdot \dfrac{1}{\sqrt{4+x} + 2}$

$= \dfrac{1}{3} \cdot \dfrac{1}{\sqrt{4+0} + 2}$

$= \dfrac{1}{3} \cdot \dfrac{1}{2+2}$

$= \dfrac{1}{12}$

Step 2 :

$f(0) = 1/4$ ……… given

Step 3 :

$f(0) \neq \lim\limits_{x \to 0} f(x)$; f is discontinuous at $x = 0$

Step 4 :

Removable Discontinuity

f can be made continuous at x = 0 by redefining it as

$$f(x) = \frac{\sqrt{4+x}-2}{3x} \quad ; \quad x \neq 0$$

$$= 1/12 \quad ; \quad x = 0$$

02. $$f(x) = \left(1 + \frac{5x}{2}\right)^{2/x} \quad ; \quad x \neq 0$$

$$= e^{5/2} \quad ; \quad x = 0$$

Discuss continuity at x = 0

Solution :

Step 1

$\lim_{x \to 0} f(x)$

$$= \lim_{x \to 0} \left(1 + \frac{5x}{2}\right)^{\frac{2}{x}}$$

$$= \lim_{x \to 0} \left(\left(1 + \frac{5x}{2}\right)^{\frac{2}{5x}}\right)^{5}$$

$$= e^{5}$$

Step 2 :

$f(0) = e^{5/2}$……… given

Step 3 :

$f(0) \neq \lim_{x \to 0} f(x)$

$\therefore$ f is disscontinuous at x = 0

Step 4 :

Removable Discontinuity

f can be made continuous at x = 0 by redefining it as

$$f(x) = \left(1 + \frac{5x}{2}\right)^{2/x} \quad ; \quad x \neq 0$$

$$= e^{5} \quad ; \quad x = 0$$

04. $f(x) = \dfrac{e^{2x} - 1}{5x}$; $x \neq 0$

$= 2$; $x = 0$ Discuss continuity at $x = 0$

Solution :

Step 1

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \dfrac{e^{2x} - 1}{5x}$

$= \lim_{x \to 0} \dfrac{1}{5} \dfrac{e^{2x} - 1}{x}$

$= \lim_{x \to 0} \dfrac{2}{5} \dfrac{e^{2x} - 1}{2x}$

$= \dfrac{2.\log e}{5}$

$= 2/5$

Step 2 :

$f(0) = 2$ ……… given

Step 3 :

$f(0) \neq \lim_{x \to 0} f(x)$

$\therefore f$ is discontinuous at $x = 0$

Step 4 :

Removable Discontinuity

f can be made continuous at $x = 0$ by redefining it as

$f(x) = \dfrac{e^{2x} - 1}{5x}$; $x \neq 0$

$= 2/5$; $x = 0$

04. $y = \tan^{-1}\left(\frac{6x}{1-5x^2}\right)$ Find dy/dx

Solution

$$y = \tan^{-1}\left(\frac{5x + x}{1 - 5x.x}\right)$$

$$y = \tan^{-1} 5x + \tan^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{1+25x^2} . \frac{d}{dx}(5x) + \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{5}{1+25x^2} + \frac{1}{1+x^2}$$

05. $y = x^{\tan^{-1}x}$ Find dy/dx

Solution

Taking log on both sides

$$\text{Log } y = \tan^{-1}x . \log x$$

Differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = \tan^{-1}x. \frac{d}{dx}\log x + \log x \frac{d}{dx}\tan^{-1}x$$

$$\frac{1}{y}\frac{dy}{dx} = \tan^{-1}x . \frac{1}{x} + \log x . \frac{1}{1+x^2}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\text{tain}^{-1}x}{x} + \frac{\log x}{1+x^2}$$

$$\frac{dy}{dx} = y\left(\frac{\tan^{-1}x}{x} + \frac{\log x}{1+x^2}\right)$$

$$\frac{dy}{dx} = x^{\tan^{-1}x}\left(\frac{\tan^{-1}x}{x} + \frac{\log x}{1+x^2}\right)$$

06. Find $\frac{dy}{dx}$, if $x^3 + y^2 + xy = 10$

Solution

$$x^3 + y^2 + xy = 10$$

differentiating wrt x

$$3x^2 + 2y \frac{dy}{dx} + x \frac{dy}{dx} + y \frac{d}{dx}(x) = 0$$

$$3x^2 + 2y \frac{dy}{dx} + x \frac{dy}{dx} + y = 0$$

$$(2y + x) \frac{dy}{dx} = -3x^2 - y$$

$$\frac{dy}{dx} = \frac{-3x^2 - y}{x + 2y}$$

Q2. Attempt any FOUR of the following (3 marks each) (12 marks)

Q2

01. $f(x) = \dfrac{35^x - 7^x - 5^x + 1}{x \cdot \log(1 + 3x)}$; $x \neq 0$

$= k$; $x = 0$ find k if f is continuous at x = 0

Solution :

Step 1

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \dfrac{35^x - 7^x - 5^x + 1}{x \cdot \log(1 + 3x)}$

$= \lim_{x \to 0} \dfrac{(7.5)^x - 7^x - 5^x + 1}{x \cdot \log(1 + 3x)}$

$= \lim_{x \to 0} \dfrac{7^x.5^x - 7^x - 5^x + 1}{x \cdot \log(1 + 3x)}$

$= \lim_{x \to 0} \dfrac{7^x(5^x - 1) - 1(5^x - 1)}{x \cdot \log(1 + 3x)}$

$= \lim_{x \to 0} \dfrac{(7^x - 1).(5^x - 1)}{x \cdot \log(1 + 3x)}$

Dividing Numerator & Denominator by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$= \lim_{x \to 0} \dfrac{\dfrac{(7^x - 1).(5^x - 1)}{x^2}}{\dfrac{x \cdot \log(1 + 3x)}{x^2}}$

$= \lim_{x \to 0} \dfrac{\dfrac{7^x - 1}{x} \cdot \dfrac{5^x - 1}{x}}{\dfrac{\log(1 + 3x)}{x}}$

$= \lim_{x \to 0} \dfrac{\dfrac{7^x - 1}{x} \cdot \dfrac{5^x - 1}{x}}{3 \; \dfrac{\log(1 + 3x)}{3x}}$

$= \dfrac{\log 7 \cdot \log 5}{3(1)}$

$= \dfrac{\log 7 \cdot \log 5}{3}$

Step 2

$f(0) = k$ ……… given

Step 3

Since f is continuous at $x = 0$

$f(0) = \lim_{x \to 0} f(x)$

$k = \dfrac{\log 7 \cdot \log 5}{3}$

02. find a & b if f(x) is continuous at x = 0 & f(1) = 2 where ;

$f(x) = x^3 + a + b \quad ; \quad x \geq 0$

$\quad = 2\sqrt{x^3 + 1} + a \quad ; \quad x < 0$

Solution :

Step 1

$\lim_{x\to0+} f(x)$

$= \lim_{x\to0} x^3 + a + b$

$= 0^2 + a + b = a + b$

Step 2

$\lim_{x\to0-} f(x)$

$= \lim_{x\to0} 2\sqrt{x^3 + 1} + b$

$= 2\sqrt{0^3 + 1} + b$

$= 2 + b$

Step 3

$f(0) = 0^2 + a + b$

$\quad = a + b$

Step 4

Since f is continuous at x = 0

$\lim_{x\to0-} f(x) = \lim_{x\to0+} f(x) = f(0)$

$2 + b = a + b = a + b$

$2 + b = a + b$

$a = 2$

Step 5

$f(1) = 2$

$1^2 + a + b = 2$

$a + b = 1$

Sub a = 2

$2 + b = 1$

$b = -1$

03. $x^3y^5 = (x + y)^8$. Show that : $\frac{dy}{dx} = \frac{y}{x}$

$3 \log x + 5 \log y = 8 \log (x + y)$

Differentiating wrt x

$$3 \frac{1}{x} + 5 \frac{1}{y} \frac{dy}{dx} = 8 \frac{1}{x + y} \frac{d}{dx} (x + y)$$

$$\frac{3}{x} + \frac{5}{y} \frac{dy}{dx} = \frac{8}{x + y} \left(1 + \frac{dy}{dx}\right)$$

$$\frac{3}{x} + \frac{5}{y} \frac{dy}{dx} = \frac{8}{x + y} + \frac{8}{x + y} \frac{dy}{dx}$$

$$\left(\frac{5}{y} - \frac{8}{x + y}\right) \frac{dy}{dx} = \frac{8}{x + y} - \frac{3}{x}$$

$$\frac{5x + 5y - 8y}{y.(x + y)} \frac{dy}{dx} = \frac{8x - 3x - 3y}{x(x + y)}$$

$$\frac{5x - 3y}{y.(x + y)} \frac{dy}{dx} = \frac{5x - 3y}{x(x + y)}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

04. Find $\frac{dy}{dx}$ if $y = x^x + 5^x$

Solution

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dv}$ ……….. (1)

Now

$u = x^x$

Taking log on both sides

$\text{Log } u = x . \log x$

$\frac{1}{u} \frac{du}{dx} = x \frac{d}{dx} \log x + \log x \frac{d}{dx} x$

$\frac{1}{u} \frac{du}{dx} = x . \frac{1}{x} + \log x . 1$

$\frac{1}{u} \frac{du}{dx} = 1 + \log x$

$\frac{du}{dx} = u (1 + \log x)$

$v = 5^x$

Differentiating wrt x

$\frac{dv}{dx} = 5^x.\log 5$

Hence

$\frac{dy}{dx} = x^x (1 + \log x) + 5^x.\log 5$

05. Differentiate : $\log(1 + x^2)$ wrt $\tan^{-1}x$

Solution

$u = \log(1 + x^2)$

Diff wrt x

$$\frac{du}{dx} = \frac{1}{1 + x^2} \cdot \frac{d}{dx}(1 + x^2)$$

$$\frac{du}{dx} = \frac{2x}{1 + x^2}.$$

$v = \cot^{-1}x$

$$\frac{dv}{dx} = \frac{-1}{1 + x^2}$$

Now

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

$$= \frac{\frac{2x}{1 + x^2}}{\frac{-1}{1 + x^2}}$$

$$= -2x$$

Q3. Attempt any TWO of the following (4 marks each) (8 marks)

Q3

01. find a & b if f(x) is continuous at x = 0

$$f(x) = \frac{e^{2x} - 1}{ax} \quad ; \quad x < 0 , a \neq 0$$

$$= 1 \quad ; \quad x = 0$$

$$= \frac{\log (1 + 7x)}{bx} \quad ; \quad x > 0 , b \neq 0$$

Solution :

Step 1

$$\lim_{x \to 0-} f(x)$$

$$= \lim_{x \to 0} \frac{e^{2x} - 1}{ax}$$

$$= \lim_{x \to 0} \frac{1}{a} \frac{e^{2x} - 1}{x}$$

$$= \lim_{x \to 0} \frac{2}{a} \frac{e^{2x} - 1}{2x}$$

$$= \frac{2.\log e}{a}$$

$$= 2/a$$

Step 2

$$\lim_{x \to 0+} f(x)$$

$$= \lim_{x \to 0} \frac{\log (1 + 7x)}{bx}$$

$$= \lim_{x \to 0} \frac{1}{b} \frac{\log (1 + 7x)}{x}$$

$$= \lim_{x \to 0} \frac{7}{b} \frac{\log (1 + 7x)}{7x}$$

$$= \frac{7}{b} (1)$$

$$= \frac{7}{b}$$

Step 3

f(0) = 1 ……… given

Step 4

Since f is continuous at x = 0

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$$

$$\frac{2}{a} = \frac{7}{b} = 1$$

$\therefore$ a = 2 & b = 7

02. Discuss the continuity of the function f at x = 0

where $f(x) = \dfrac{5^x + 5^{-x} - 2}{\cos 2x - \cos 6x}$; $x \neq 0$

$= \dfrac{1}{8}(\log 5)^2$; $x = 0$

Solution :

Step 1

$$\lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \frac{5^x + 5^{-x} - 2}{\cos 2x - \cos 6x}$$

$$= \lim_{x \to 0} \frac{5^x + \frac{1}{5^x} - 2}{-2 \sin \frac{2x+6x}{2} . \sin \frac{2x-6x}{2}}$$

$$= \lim_{x \to 0} \frac{\frac{(5^x)^2 + 1 - 2.5^x}{5^x}}{-2 \sin \frac{8x}{2} . \sin \frac{-4x}{2}}$$

$$= \lim_{x \to 0} \frac{\frac{(5^x - 1)^2}{5^x}}{-2 \sin 4x . \sin -2x}$$

$$= \lim_{x \to 0} \frac{\frac{(5^x - 1)^2}{5^x}}{2 \sin 4x . \sin 2x}$$

Dividing Numerator & Denominator by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$$= \lim_{x \to 0} \frac{\frac{(5^x - 1)^2}{x^2 5^x}}{\frac{2 \sin 4x . \sin 2x}{x^2.}}$$

$$= \lim_{x \to 0} \frac{\left(\frac{5^x - 1}{x}\right)^2 \frac{1}{5^x}}{2 \frac{\sin 4x}{x.} . \frac{\sin 2x}{x}}$$

$$= \lim_{x \to 0} \frac{\left(\frac{5^x - 1}{x}\right)^2 \frac{1}{5^x}}{2 .4 \frac{\sin 4x}{4x .} . 2 \frac{\sin 2x}{2x}}$$

$$= \frac{(\log 5)^2}{2.4.(1).2.(1)}$$

$$= \frac{(\log 5)^2}{16}$$

Step 2 :

$f(0) = \dfrac{(\log 5)^2}{8}$ ……… given

Step 3 :

$f(0) \neq \lim_{x \to 0} f(x)$

$\therefore$ f is discontinuous at x = 0

Step 4 :

Removable Discontinuity

f can be made continuous at x = 0 by redefining it as

$f(x) = \dfrac{5^x + 5^{-x} - 2}{\cos 2x - \cos 6x}$; $x \neq 0$

$= \dfrac{(\log 5)^2}{16}$; $x = 0$

03. If $x^y = e^{x-y}$; show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Solution

$x^y = e^{x-y}$

Taking log on both sides

$y \cdot \log x = (x - y) \cdot \log e$

$y \cdot \log x = x - y$

$y \cdot \log x + y = x$

$y(\log x + 1) = x$

$y = \frac{x}{1 + \log x}$

cont.

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$\frac{dy}{dx} = \frac{(1 + \log x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \log x)^2}$$

$$\frac{dy}{dx} = \frac{1 + \log x - 1}{(1 + \log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$$ ………. proved